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ANALYTICAL SOLUTION OF HEAT-CONDUCTION PROBLEMS WITH A VARIABLE INITIAL CONDITION BASED ON DETERMINING A TEMPERATURE PERTURBATION FRONT

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With the aid of the integral heat-balance method an analytical solution of the problem of unsteady-state heat conduction has been obtained for an infinite plate with a variable initial condition. To increase the accuracy of solution by the integral method additional boundary conditions are introduced which are determined from the initial differential equation and basic boundary conditions, including those prescribed at the temperature perturbation front.

Introduction. Among approximate analytical methods there are those into which the notion of the temperature perturbation front is introduced. With such an approach, the process of heating (cooling) is formally divided into two stages: the first is characterized by the gradual motion of the temperature perturbation front from the surface to the center of a body and the second — by the change in the temperature over the entire volume of the body until the development of a steady state. The finite velocity of the temperature perturbation front motion is taken into account by introducing a new function $q_1(\mathrm{Fo})$ called the penetration depth (the depth of a thermal layer) [1–6]. The merit of these methods is the possibility of obtaining simple (in form) analytical solutions for both regular and irregular heat conduction processes. Among the disadvantages there is the necessity of a prior selection of the coordinate dependence for the temperature function sought. In the majority of the works mentioned above a quadratic or cubic parabola is taken to represent the temperature profile. Such ambiguity of the solution leads to the problem of its accuracy, since, by adopting beforehand a particular profile, each time we will obtain different results.

The obvious way for increasing the accuracy of solution is approximation of the temperature function by higher-degree polynomials. However, to determine their unknown coefficients their initial boundary conditions turn out to be insufficient. In view of this, one has to employ additional boundary conditions. In the present work, such conditions are obtained from the initial differential equation with the use of basic boundary conditions and the conditions prescribed at the temperature perturbation front.

Statement of the Problem. As a specific example, we consider a boundary-value problem of unsteady-state heat conduction with linear distribution of initial temperature over the plate thickness. The mathematical statement of the problem in this case has the form

$$\frac{\partial T(x,\tau)}{\partial \tau} = a \frac{\partial^2 T(x,\tau)}{\partial x^2}, \quad \tau > 0, \quad 0 < x < \delta ; \tag{1}$$

$$T(x, 0) = T_0 - \frac{x}{\delta}(T_0 - T_{\mathrm{w}}) ; \tag{2}$$

$$T(0, \tau) = T(\delta, \tau) = T_{\mathrm{w}} . \tag{3}$$

We introduce the following dimensionless quantities:

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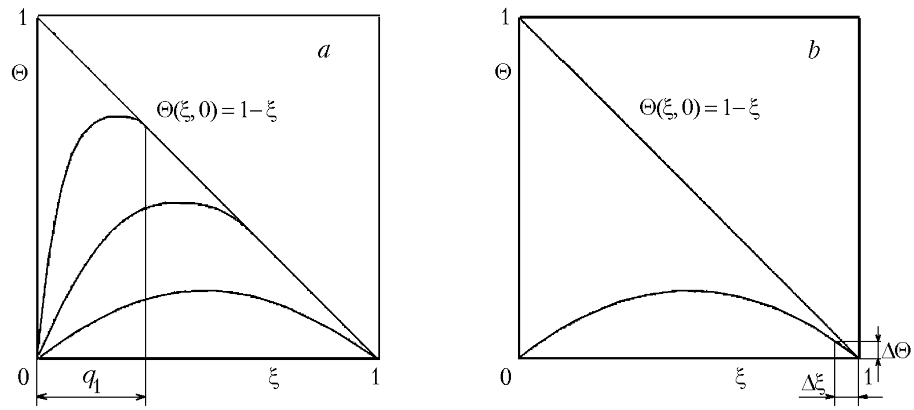

Fig. 1. Calculation scheme of heat transfer.

$$\Theta = \frac{T(x, \tau) - T_w}{T_0 - T_w}, \quad \xi = \frac{x}{\delta}, \quad \text{Fo} = \frac{a\tau}{\delta^2}.$$

In view of the notation adopted, problem (1)–(3) takes the form

$$\frac{\partial \Theta(\xi, \text{Fo})}{\partial \text{Fo}} = \frac{\partial^2 \Theta(\xi, \text{Fo})}{\partial \xi^2}, \quad \text{Fo} > 0, \quad 0 < \xi < 1; \tag{4}$$

$$\Theta(\xi, 0) = 1 - \xi; \tag{5}$$

$$\Theta(0, \text{Fo}) = 0; \tag{6}$$

$$\Theta(1, \text{Fo}) = 0. \tag{7}$$

The process of heat transfer will be divided into two stages in time: 1) $0 \le \text{Fo} \le \text{Fo}_1$, where Fo_1 is the time needed by the temperature perturbation front to reach the coordinate $\xi = 1$; 2) $\text{Fo}_1 \le \text{Fo} < \infty$, where heat transfer occupies the entire volume of the body till a stationary regime is attained.

To fix the temperature perturbation front, we will introduce a boundary moving in time and separating the initial region $0 \le \xi \le 1$ into two subregions: $0 \le \xi \le q_1(\text{Fo})$ and $q_1(\text{Fo}) \le \xi \le 1$, where $q_1(\text{Fo})$ is the function which determines the advancement of the interface along the coordinate ξ depending on time. Note that in the region located beyond the temperature perturbation front the initial temperature is preserved (Fig. 1a). The first stage of the process terminates as soon as the position $q_1(\text{Fo}) = 1$ is reached by the moving boundary.

In view of the fact that the new function $q_1(\text{Fo})$ has been introduced into consideration the conditions fulfilled at the temperature perturbation front should be added. They are found from initial condition (5) and have the form

$$\Theta(\xi, \text{Fo})\Big|_{\xi=q_1} = 1 - q_1, \quad \frac{\partial \Theta(\xi, \text{Fo})}{\partial \xi}\Big|_{\xi=q_1} = -1. \tag{8}$$

The first condition of (8) means the equality of temperature at the temperature perturbation front to the initial temperature of the body, whereas the second condition shows that the line of the temperature profile is tangent to the line of the initial temperature $\Theta(\xi, 0) = 1 - \xi$ (see Fig. 1a).

Thus, for the first stage of the process one has to solve Eq. (4) with boundary conditions (6) and (8). Here, boundary condition (7) is not needed, since it does not influence the process of heat transfer in its first stage.

Now impose the requirement that the sought-for solution of problem (4)–(6), (8) could satisfy not the initial equation (4), but rather that averaged over the thermal layer thickness. For this purpose, we take an integral of Eq. (4) along the coordinate ξ within the limits from zero to $q_1(\text{Fo})$. This yields an integral condition (heat balance integral),

$$\int_0^{q_1(\text{Fo})} \frac{\partial \Theta(\xi, \text{Fo})}{\partial \text{Fo}} d\xi = \int_0^{q_1(\text{Fo})} \frac{\partial^2 \Theta(\xi, \text{Fo})}{\partial \xi^2} d\xi . \tag{9}$$

Having determined the integral on the right-hand side of relation (9), we find the final expression for the heat balance integral

$$\int_0^{q_1(\text{Fo})} \frac{\partial \Theta(\xi, \text{Fo})}{\partial \text{Fo}} d\xi = \left.\frac{\partial \Theta(\xi, \text{Fo})}{\partial \xi}\right|_{\xi=q_1} - \left.\frac{\partial \Theta(\xi, \text{Fo})}{\partial \xi}\right|_{\xi=0} . \tag{10}$$

The solution of problem (4)–(6), (8) is sought in the form of the polynomial

$$\Theta(\xi, \text{Fo}) = \sum_{k=0}^{n} a_k(q_1)\, \xi^k . \tag{11}$$

In order to find the solution of problem (4)–(8) in the first approximation we substitute Eq. (11), limiting ourselves by three terms of the series, into boundary conditions (6) and (8). From this, to determine the unknown coefficients a_k ($k = 0, 1, 2$) we obtain a system of three algebraic linear equations. Relation (11), with allowance for the coefficients a_k found from the solution of this system, will take the form

$$\Theta(\xi, \text{Fo}) = 2\frac{\xi}{q_1} - \frac{\xi^2}{q_1^2} - \xi . \tag{12}$$

Having substituted Eq. (12) into the heat balance integral (10), we come to an ordinary differential equation for the unknown function $q_1(\text{Fo})$;

$$q_1 \frac{dq_1}{d\text{Fo}} = 6 . \tag{13}$$

Dividing the variables in (13) and intigrating, with the initial condition $q_1(0) = 0$ we obtain

$$q_1(\text{Fo}) = \sqrt{12\text{Fo}} . \tag{14}$$

Assuming in Eq. (14) that $q_1 = 1$, we find the time of termination of the first stage of the process $\text{Fo} = \text{Fo}_1 = 0.0833333$. The results of calculations by Eq. (12) (Fig. 2) differ from the temperature values obtained by the pivot method of [7] by no more than 4%.

To raise the accuracy of analytical solution one has to increase the number of series terms in Eq. (11). However, when the number of coefficients $a_k(q_1)$ exceeds 3, additional boundary conditions should be used to determine them [6, 8, 9]. For this purpose, we will successively differentiate boundary conditions (6), (8) with respect to the variable Fo and Eq. (4) — to variable ξ. So that the first additional boundary condition could be obtained, we differentiate Eq. (6) with respect to Fo:

$$\frac{\partial \Theta(0, \text{Fo})}{\partial \text{Fo}} = 0 . \tag{15}$$

We write Eq. (4) for the point $\xi = 0$:

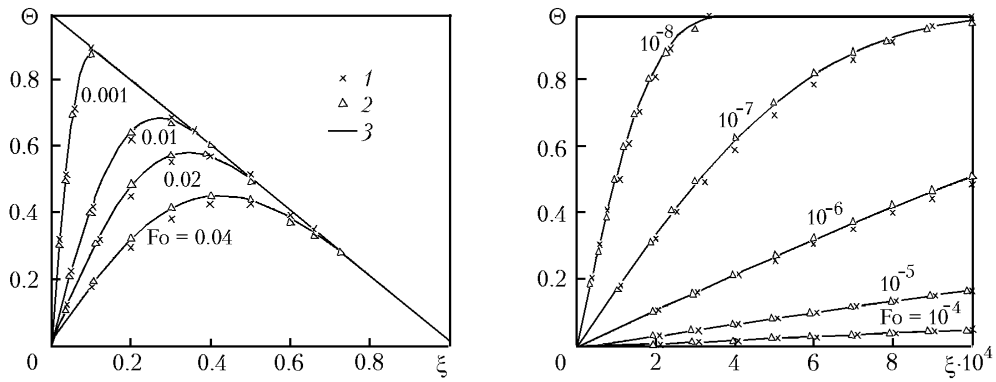

Fig. 2. Distribution, in the plate, of dimensionless temperature calculated: 1) from Eq. (12) (a first approximation); 2) by the pivot method [9]; 3) from Eq. (26) (a second approximation).

$$\frac{\partial \Theta\,(0, \mathrm{Fo})}{\partial \mathrm{Fo}} = \left. \frac{\partial^2 \Theta\,(\xi, \mathrm{Fo})}{\partial \xi^2} \right|_{\xi=0} . \tag{16}$$

Comparing Eqs. (15) and (16), we obtain the first additional boundary condition:

$$\left. \frac{\partial^2 \Theta\,(\xi, \mathrm{Fo})}{\partial \xi^2} \right|_{\xi=0} = 0 . \tag{17}$$

To obtain the second additional boundary condition, we differentiate the first relation of (8) with respect to Fo:

$$\left. \frac{\partial \Theta\,(\xi, \mathrm{Fo})}{\partial \mathrm{Fo}} \right|_{\xi=q_1} + \left. \frac{\partial \Theta\,(\xi, \mathrm{Fo})}{\partial \xi} \right|_{\xi=q_1} \frac{dq_1}{d\mathrm{Fo}} = -\frac{dq_1}{d\mathrm{Fo}} . \tag{18}$$

We write Eq. (4) for $\xi = q_1(\mathrm{Fo})$:

$$\frac{\partial \Theta\,(q_1, \mathrm{Fo})}{\partial \mathrm{Fo}} = \left. \frac{\partial^2 \Theta\,(\xi, \mathrm{Fo})}{\partial \xi^2} \right|_{\xi=q_1} . \tag{19}$$

Comparing Eqs. (18) and (19), subject to the second relation of (8), we obtain the second additional boundary condition:

$$\left. \frac{\partial^2 \Theta\,(\xi, \mathrm{Fo})}{\partial \xi^2} \right|_{\xi=q_1} = 0 . \tag{20}$$

In order to obtain the third additional boundary condition we differentiate the second relation of (8) with respect to Fo:

$$\left. \frac{\partial^2 \Theta\,(\xi, \mathrm{Fo})}{\partial \mathrm{Fo}\,\partial \xi} \right|_{\xi=q_1} = 0 . \tag{21}$$

We differentiate Eq. (4) with respect to ξ and write the resulting relation for the point $\xi = q_1(\mathrm{Fo})$:

$$\left. \frac{\partial^2 \Theta\,(\xi, \mathrm{Fo})}{\partial \mathrm{Fo}\,\partial \xi} \right|_{\xi=q_1} = \left. \frac{\partial^3 \Theta\,(\xi, \mathrm{Fo})}{\partial \xi^3} \right|_{\xi=q_1} . \tag{22}$$

Comparing Eqs. (21) and (22), we obtain the third additional boundary condition:

$$\left.\frac{\partial^3\Theta(\xi,\mathrm{Fo})}{\partial\xi^3}\right|_{\xi=q_1}=0\,. \tag{23}$$

With the aid of the additional boundary conditions (17), (20), (23) and prescribed conditions (6) and (8) we may determine already six coefficients of polynomial (11). Substituting (11), limiting ourselves to six terms of the series, into all of the above-given boundary conditions, we obtain the following system of algebraic linear equations for the coefficients a_k ($k = \overline{0,5}$):

$$\left(a_0+a_1\xi+a_2\xi^2+a_3\xi^3+a_4\xi^4+a_5\xi^5\right)_{\xi=0}=0\,,\quad a_0+a_1q_1+a_2q_1^2+a_3q_1^3+a_4q_1^4+a_5q_1^5=1-q_1\,,$$

$$a_1+2a_2q_1+3a_3q_1^2+4a_4q_1^3+5a_5q_1^5=-1\,,\quad \left(2a_2+6a_3\xi+12a_4\xi^2+20a_5\xi^3\right)_{\xi=0}=0\,,$$

$$2a_2+6a_3q_1+12a_4q_1^2+20a_5q_1^3=0\,,\ 6a_3+24a_4q_1+60a_5q_1^2=0\,. \tag{24}$$

It follows from the first and fourth equations of system (24) that $a_0 = 0$ and $a_2 = 0$; for the remaining coefficients a_k its solution yields

$$a_1=-(2q_1-5)/2q_1\,,\quad a_3=-5/q_1^3\,,\quad a_4=5/q_1^4\,,\quad a_5=-3/2q_1^5\,. \tag{25}$$

Relation (11), with allowance for the obtained values of coefficients a_k ($k = \overline{0,5}$), takes the form

$$\Theta(\xi,\mathrm{Fo})=1-\xi-\left(1+\frac{3}{2}\frac{\xi}{q_1}\right)\left(1-\frac{\xi}{q_1}\right)^4. \tag{26}$$

Substitution of Eq. (26) into the heat balance integral (10) yields an ordinary differential equation for the unknown function $q_1(\mathrm{Fo})$:

$$q_1\frac{dq_1}{d\mathrm{Fo}}=10\,. \tag{27}$$

Separating the variables in Eq. (27) and integrating, under the initial condition $q_1(0) = 0$, we obtain

$$q_1(\mathrm{Fo})=\sqrt{20\mathrm{Fo}}\,. \tag{28}$$

Assuming in Eq. (28) that $q_1 = 1$, we find the time of termination of the first stage of the process in the second approximation $\mathrm{Fo} = \mathrm{Fo}_1 = 0.05$.

Relation (26) and (27) determine the solution of problem (4)–(8) in the second approximation of the first stage of the process. The results of calculations by Eq. (26) and comparison of these results with those obtained by the pivot method of [9] are presented in Fig. 2. We may conclude that the temperatures given by Eq. (26) differ from those obtained by the pivot method by no more than 0.5%.

Figure 3 presents the graph of the dependence of the temperature perturbation front on dimensionless time. Its analysis shows that the most intense mixing of this front along the coordinate ξ occurs during a very short initial period of time. In particular, the time needed for the moving boundary to attain the coordinate $\xi = 0.05$ is $\mathrm{Fo} = 1.25\cdot10^{-4}$. From this we find the average velocity of the temperature perturbation front motion within the range $0 \le \xi \le 0.05$. We take the following initial data: $a = 12.5\cdot10^{-6}$ m^2/sec; $\delta = 0.01$ m. The dimensionless distance $\xi = 0.05$ in a dimensional form corresponds to $x = \xi\delta = 0.05\cdot0.01 = 5\cdot10^{-4}$ m. This is the distance traversed by the temperature perturbation front in time $\tau = \mathrm{Fo}\delta^2/a = 1\cdot10^{-3}$ sec. From this the average velocity of motion is $v_{\mathrm{av}} = x/\tau = 0.5$ m/sec. The average velocity within the ranges $0 \le \xi \le 0.01$ ($\mathrm{Fo} = 5\cdot10^{-6}$) and $0 \le \xi \le 0.001$ ($\mathrm{Fo} = 5\cdot10^{-8}$) will be equal to $v_{\mathrm{av}} = 2.5$ and 25 m/sec, respectively. Consequently, with a decrease in the considered range along the coor-

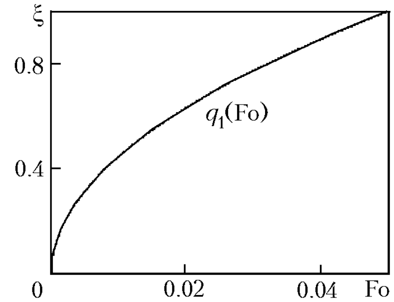

Fig. 3. Motion of the temperature perturbation front along the coordinate ξ depending on the dimensionless time Fo.

dinate the average velocity of the temperature perturbation front motion increases substantially and it tends to infinity when $\Delta x \to 0$.

The second stage of the thermal process, which corresponds to the time $\mathrm{Fo} \geq \mathrm{Fo}_1$, is characterized by temperature variations already over the entire section of the plate till a stationary regime is attained. At this stage the notion of a thermal layer loses its meaning, and as a generalized coordinate we take the scalar value of the temperature gradient on the plate surface (see Fig. 1b):

$$\mathrm{grad}\,\Theta\,(1, \mathrm{Fo}) = \frac{\partial\Theta\,(1, \mathrm{Fo})}{\partial\xi} = q_2\,(\mathrm{Fo})\,. \tag{29}$$

In this case, the mathematical statement of the problem has the form

$$\frac{\partial\Theta\,(\xi, \mathrm{Fo})}{\partial\mathrm{Fo}} = \frac{\partial^2\Theta\,(\xi, \mathrm{Fo})}{\partial\xi^2}\,, \quad \mathrm{Fo} > \mathrm{Fo}_1\,, \quad 0 < \xi < 1\,; \tag{30}$$

$$\Theta\,(\xi, \mathrm{Fo}_1) = (1 - \xi)\,\xi\,; \tag{31}$$

$$\Theta\,(0, \mathrm{Fo}) = 0\,; \quad \Theta\,(1, \mathrm{Fo}) = 0\,. \tag{32}$$

As the initial condition we adopt the distribution of temperature at the end of the first stage of the process (the first approximation, see Eq. (12) at $q_1(\mathrm{Fo}_1) = 1$).

In view of the fact that the new function $q_2(\mathrm{Fo})$ is introduced into consideration, one has to add the corresponding boundary condition:

$$\frac{\partial\Theta\,(1, \mathrm{Fo})}{\partial\xi} = q_2\,(\mathrm{Fo})\,. \tag{33}$$

The heat balance integral for the second stage of the thermal process has the form

$$\int_0^1 \frac{\partial\Theta\,(\xi, \mathrm{Fo})}{\partial\mathrm{Fo}}\,d\xi = \frac{\partial\Theta\,(1, \mathrm{Fo})}{\partial\xi} - \frac{\partial\Theta\,(0, \mathrm{Fo})}{\partial\xi} = q_2\,(\mathrm{Fo}) - \frac{\partial\Theta\,(0, \mathrm{Fo})}{\partial\xi}\,. \tag{34}$$

Just as for the first stage, the sought-for temperature profile will be represented as a polynomial of degree n:

$$\Theta\,(\xi, \mathrm{Fo}) = \sum_{k=0}^{n} b_k\,(q_2)\,\xi^k\,. \tag{35}$$

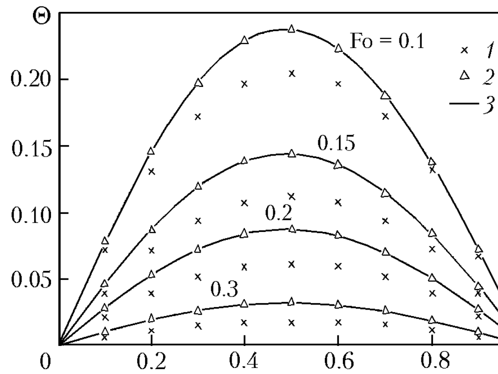

Fig. 4. Distribution, in the plate, of dimensionless temperature calculated: 1) from Eq. (41) (a first approximation); 2) by the pivot method [9]; 3) from Eq. (55) (a second approximation).

To find the solution of problem (30)–(33) in the first approximation we substitute Eq. (35) (limiting ourselves to three terms of the series) into boundary conditions (32) and (33). From this, to determine the unknown coefficients b_k ($k = 0, 1, 2$), we obtain a system of three algebraic linear equations. Subject to the coefficients b_k found from the solution of this system, the relation (35) takes the form

$$\Theta(\xi, \mathrm{Fo}) = q_2 \xi (\xi - 1) . \tag{36}$$

Substituting Eq. (36) into the heat balance integral (34), for the unknown function $q_2(\mathrm{Fo})$ we write the ordinary differential equation

$$\frac{dq_2}{d\mathrm{Fo}} = - 12 q_2 . \tag{37}$$

Dividing the variables in Eq. (37) and integrating we obtain

$$q_2(\mathrm{Fo}) = C \exp[- 12\mathrm{Fo}] . \tag{38}$$

The integration constant C is determined from the initial condition of the form

$$q_2(\mathrm{Fo}_1) = \left. \frac{\partial \Theta(\xi, \mathrm{Fo}_1)}{\partial \xi} \right|_{\xi=1} = - 1 . \tag{39}$$

Relation (39) is obtained from Eq. (12) at $\mathrm{Fo} = \mathrm{Fo}_1$, $q_1 = 1$, and $\xi = 1$. After the integration constant is defined, relation (38) takes the form

$$q_2(\mathrm{Fo}) = - \exp[- 12 (\mathrm{Fo} - \mathrm{Fo}_1)] . \tag{40}$$

Substituting Eq. (40) into Eq. (36), we find the solution of problem (30)–(33) in the first approximation of the second stage of the process:

$$\Theta(\xi, \mathrm{Fo}) = \xi (1 - \xi) \exp[- 12 (\mathrm{Fo} - \mathrm{Fo}_1)] . \tag{41}$$

By performing direct substitution, we can see that relation (41) exactly satisfies the heat balance integral (34), boundary conditions (32) and (33), as well as the initial condition (31). The results of calculations by Eq. (41), as well as by the pivot method, are presented in Fig. 4. Their analysis allows the conclusion that the maximum difference of the results obtained from Eq. (41) from the temperature values found by the pivot method does not exceed 8%.

For Fo_1 we took the value $\mathrm{Fo}_1 = 0.0833333$ obtained in the first approximation of the first stage of the process. To raise the accuracy, we find the solution of problem (30)–(33) in the second approximation employing additional boundary conditions. To obtain them, we differentiate boundary conditions (32) and (33) with respect to the variable Fo:

$$\frac{\partial \Theta\,(0, \mathrm{Fo})}{\partial \mathrm{Fo}} = 0\,, \quad \frac{\partial \Theta\,(1, \mathrm{Fo})}{\partial \mathrm{Fo}} = 0\,, \quad \frac{\partial^2 \Theta\,(1, \mathrm{Fo})}{\partial \mathrm{Fo}\,\partial \xi} = \frac{dq_2}{d\mathrm{Fo}}\,. \tag{42}$$

Writing Eq. (30) for the point $\xi = 0$ and comparing the resulting expression with the first relation from (42), we obtain the first additional boundary condition:

$$\left.\frac{\partial^2 \Theta\,(\xi, \mathrm{Fo})}{\partial \xi^2}\right|_{\xi=0} = 0\,. \tag{43}$$

Comparison of Eq. (30) at the point $\xi = 1$ with the second relation from (42) allows us to formulate the second additional boundary condition:

$$\left.\frac{\partial^2 \Theta\,(\xi, \mathrm{Fo})}{\partial \xi^2}\right|_{\xi=1} = 0\,. \tag{44}$$

We differentiate Eq. (30) with respect to the variable ξ and write the resulting relation for the point $\xi = 1$:

$$\left.\frac{\partial^2 \Theta\,(\xi, \mathrm{Fo})}{\partial \mathrm{Fo}\,\partial \xi}\right|_{\xi=1} = \left.\frac{\partial^3 \Theta\,(\xi, \mathrm{Fo})}{\partial \xi^3}\right|_{\xi=1}\,. \tag{45}$$

Comparing the third relation from (42) with relation (45), we obtain the third additional boundary condition:

$$\left.\frac{\partial^3 \Theta\,(\xi, \mathrm{Fo})}{\partial \xi^3}\right|_{\xi=1} = \frac{dq_2}{d\mathrm{Fo}}\,. \tag{46}$$

Having substituted Eq. (35), limiting ourselves to six terms of the series, into boundary conditions (32), (33), (43), (44), and (46), for the unknown coefficients b_k ($k = \overline{0, 5}$) we obtain the following system of algebraic linear equations:

$$\begin{gathered}\left(b_0 + b_1\xi + b_2\xi^2 + b_3\xi^3 + b_4\xi^4 + b_5\xi^5\right)_{\xi=0} = 0\,, \quad b_0 + b_1 + b_2 + b_3 + b_4 + b_5 = 0\,, \\ b_1 + 2b_2 + 3b_3 + 4b_4 + 5b_5 = q_2\,, \quad \left(2b_2 + 6b_3\xi + 12b_4\xi^2 + 20b_5\xi^3\right)_{\xi=0} = 0\,, \\ 2b_2 + 6b_3 + 12b_4 + 20b_5 = 0\,, \quad 6b_3 + 24b_4 + 60b_5 = dq_2/d\mathrm{Fo}\,.\end{gathered} \tag{47}$$

From the first and fourth equations of system (47) it follows that $b_0 = 0$; $b_2 = 0$; for the remaining coefficients its solution yields

$$\begin{gathered}b_0 = 0\,, \quad b_1 = -\frac{3}{2}\,q_2 - \frac{1}{24}\,\frac{dq_2}{d\mathrm{Fo}}\,, \quad b_2 = 0\,, \\ b_3 = 5q_2 + \frac{1}{4}\,\frac{dq_2}{d\mathrm{Fo}}\,, \quad b_4 = -\,5q_2 - \frac{1}{3}\,\frac{dq_2}{d\mathrm{Fo}}\,, \quad b_5 = \frac{3}{2}\,q_2 + \frac{1}{8}\,\frac{dq_2}{d\mathrm{Fo}}\,.\end{gathered} \tag{48}$$

Substituting the values found for the coefficients b_k ($k = \overline{0, 5}$) into Eq. (35), we obtain

$$\Theta(\xi, \mathrm{Fo}) = \frac{1}{2}\left(3\xi^5 - 10\xi^4 + 10\xi^3 - 3\xi\right) q_2 + \frac{1}{24}\left(3\xi^5 - 8\xi^4 + 6\xi^3 - \xi\right)\frac{dq_2}{d\mathrm{Fo}} . \quad (49)$$

After the substitution of Eq. (49) into the heat balance integral (34) we arrive at the following ordinary differential equation for the unknown function $q_2(\mathrm{Fo})$:

$$\frac{d^2 q_2}{d\mathrm{Fo}^2} + 70\frac{dq_2}{d\mathrm{Fo}} + 600 q_2 = 0 \quad (50)$$

with boundary conditions

$$q_2(\mathrm{Fo}_1) = -1 , \quad \left.\frac{dq_2(\mathrm{Fo}_1)}{d\mathrm{Fo}}\right|_{\mathrm{Fo}=\mathrm{Fo}_1} = 0 . \quad (51)$$

The characteristic equation of the differential equation (50) has the form

$$r^2 + 70r + 600 = 0 .$$

Its roots are $r_1 = -60$ and $r_2 = -10$. From this, the general solution of Eq. (50) is written as

$$q_2 = C_1 \exp(-60\mathrm{Fo}) + C_2 \exp(-10\mathrm{Fo}) . \quad (52)$$

The integration constants C_1 and C_2 are determined from boundary conditions (51):

$$C_1 = 0.2 \exp(60\mathrm{Fo}_1) , \quad C_2 = -1.2 \exp(10\mathrm{Fo}_1) . \quad (53)$$

Subject to (53), Eq. (52) yields

$$q_2(\mathrm{Fo}) = 0.2 \exp[-60(\mathrm{Fo} - \mathrm{Fo}_1)] - 1.2 \exp[-10(\mathrm{Fo} - \mathrm{Fo}_1)] . \quad (54)$$

Substituting Eq. (54) into Eq. (49), we find the final expression to determine the temperature in the second approximation of the second stage of the process:

$$\Theta(\xi, \mathrm{Fo}) = \left(1.3\xi - 3\xi^3 + 2\xi^4 - 0.3\xi^5\right)\exp[-10(\mathrm{Fo} - 0.05)] - \left(0.8\xi - 4\xi^3 + 5\xi^4 - 1.8\xi^5\right)\exp[-60(\mathrm{Fo} - 0.05)] . \quad (55)$$

For Fo in Eq. (55) we took the value $\mathrm{Fo}_1 = 0.05$ obtained in the second approximation of the first stage of the process (see Eq. (28) at $q_1 = 1$).

In order to derive additional boundary conditions in the subsequent approximations of both the first and second stages of the process it is necessary to differentiate the additional boundary conditions of the second approximation in the variable Fo and to differentiate twice the initial differential equation in ξ and apply it at the points $\xi = 0$ and $\xi = 1$. Comparing the resulting relations, we find the following three additional boundary conditions that allow us to determine the solution of the problem in the third approximation:

for the first stage of the process

$$\frac{\partial^4\Theta(0, \mathrm{Fo})}{\partial\xi^4} = 0 , \quad \frac{\partial^4\Theta(q_1, \mathrm{Fo})}{\partial\xi^4} = 0 , \quad \frac{\partial^5\Theta(q_1, \mathrm{Fo})}{\partial\xi^5} = 0 ;$$

for the second stage of the process

$$\frac{\partial^4 \Theta (0, \mathrm{Fo})}{\partial \xi^4} = 0 , \quad \frac{\partial^4 \Theta (1, \mathrm{Fo})}{\partial \xi^4} = 0 , \quad \frac{\partial^5 \Theta (1, \mathrm{Fo})}{\partial \xi^5} = 0 .$$

Analogously, one can obtain additional boundary conditions for subsequent approximations too.

The results of calculations by Eq. (55) and by the pivot method are presented in Fig. 4. Their analysis allows the conclusion that the temperature values given by Eq. (52) virtually coincide with those obtained by the pivot method.

CONCLUSIONS

1. Based on the integral heat balance method with the use of the notion of the temperature perturbation front and additional boundary conditions, analytical solutions have been obtained for regular and irregular processes of heat conduction in an infinite plate with a variable boundary condition.

2. A technique of finding additional boundary conditions has been suggested. It is based on the use of the initial differential equation and prescribed boundary conditions including the conditions at the temperature perturbation front. Their application allows one, with a minimum number of approximations, to much better satisfy the basic differential equation in the entire region of variation of the variable ξ $(0 \le \xi \le 1)$, since this equation, due to the satisfaction of additional boundary conditions, is fulfilled exactly at all of the points where the temperature perturbation front is located at the corresponding values of Fourier number.

NOTATION

a, thermal diffusivity, $\mathrm{m^2/sec}$; $a_k(q_1)$, $b_k(q_2)$, unknown coefficients; $q_1(\mathrm{Fo})$, $q_2(\mathrm{Fo})$, time-dependent functions; r_1, r_2, roots of characteristic equation; T, temperature, $^{\mathrm{o}}$C; T_0, initial temperature, $^{\mathrm{o}}$C; T_{w}, wall temperature, $^{\mathrm{o}}$C; v_{av}, average velocity, m/sec; x, coordinate, m; δ, plate thickness, m; Θ, relative excess temperature; ξ, dimensionless coordinate; τ, time, sec; Fo, Fourier number. Subscripts: 0, initial parameters; w, wall; av, average; ∞, parameters at infinity.

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